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A STOCK AND FLOW MODEL OF ADVERTISING RESPONSE WITH SATURATION, CARRYOVER, AND COMPETITIVE FEEDBACK

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Advertising rarely acts instantaneously. It builds a stock of goodwill that decays over time, exhibits diminishing returns at higher levels of exposure, and interacts with competitors' actions. The contribution here is a compact stock and flow formulation that joins three facts into one system: carryover, saturation, and competitive feedback. It draws on classic dynamic models of goodwill accumulation and competition while adding a bounded response map that fits modern digital data where marginal returns decline rapidly at scale [1][2][3].

Let $G(t)$ denote the firm's advertising goodwill at time t and $u(t)$ its advertising spend rate. The carryover mechanism is modeled as an adstock ordinary differential equation with linear decay $\lambda > 0$ and conversion sensitivity $\beta > 0$:

$$\dot{G}(t) = -\lambda G(t) + \beta u(t)$$

This is the continuous-time analogue of the Nerlove–Arrow goodwill equation and captures the empirical fact that advertising effects persist but fade [2].

Because response saturates, realized sales flow $Q(t)$ is linked to goodwill through a bounded, S-shaped transformation. A convenient choice that is simple and Word-compatible is the Hill form with maximum scale $S_{max} > 0$, half-saturation $\kappa > 0$, and curvature $\eta \geq 1$:

$$S(G) = S_{max}\frac{G^{\eta}}{\kappa^{\eta} + G^{\eta}}, Q(t) = S\big(G(t)\big)$$

This captures diminishing marginal returns at high exposure and the slow-start region often observed in practice. Recent MMM frameworks adopt related saturation transforms to stabilize inference and reflect true concavity [6].

Competitive feedback enters through market share dynamics. For two firms i and j, let $m_i(t)$ be firm i's share, $\alpha_i > 0$ a conversion intensity, and tie the attraction force to goodwill. A Lanchester-type specification is:

$$\dot{m}_i(t) = \alpha_i G_i(t)\big(1 - m_i(t)\big) - \alpha_j G_j(t) m_i(t)$$

This parsimoniously encodes offensive and defensive effects of advertising on share and yields closed-form or numerically stable equilibria in many cases [3].

Given exogenous baseline demand $D(t)$, price p, and unit advertising cost $c > 0$, a firm chooses $u(t)$ to maximize discounted profit with rate $r > 0$:

$$\max_{u(t)\geq 0} \int_0^T [p\, D(t)\, m_i(t) - c\, u(t)] e^{-rt}\, dt$$

The system couples the control $u(t)$ to profits via $G(t)$, $S\big(G(t)\big)$, and $m_i(t)$. Under ad fatigue the optimal $u(t)$ can be interior and decline when past exposure is high, which aligns with recent optimal-control results that incorporate negative marginal utility from over-messaging [5].

Estimation usually proceeds in discrete time. Let Δ be the interval and define the discrete adstock g_t with carryover $\delta = e^{-\lambda\Delta}$ and gain $\gamma = \beta\big(1 - e^{-\lambda\Delta}\big)$. The state update is $g_t = \delta g_{t-1} + \gamma u_t$. A saturated link from state to sales may be written as $q_t = S(g_t) + \varepsilon_t$, with ε_t an error term. Inline, the discrete pair is $g_t = \delta g_{t-1} + \gamma u_t$, $q_t = S(g_t) + \varepsilon_t$. The competitive term can be approximated by $m_{i,t} - m_{i,t-1} \approx \alpha_i g_{i,t}\big(1 - m_{i,t-1}\big) - \alpha_j g_{j,t} m_{i,t-1}$, which is estimable with nonlinear least squares or state-space filters. These specifications generalize early operations-research models that used linear sales and constant saturation thresholds [1].

Managerially, the model separates mechanism from tactic. Portfolio bid strategies in ad platforms change how $u(t)$ is delivered within channels, but they should be optimized against the structural response surface $S(G)$ and the competitive term rather than used as a proxy for demand drivers [4]. In practice, one fits λ, β, κ, and η on rolling windows, constrains η to avoid overfitting, and stress-tests results under plausible competitive trajectories. When ad fatigue is detected, the policy that reduces $u(t)$ in periods of high g_t can be profit-maximizing despite short-run volume loss [5]. The framework is compact, interpretable, and anchored in a literature that unifies goodwill carryover, saturation, and competition [2][3][6].

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